

Average position of quantum walks with an arbitrary initial state*

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Abstract

We investigated discrete time quantum walks with an arbitrary initial state $|\Psi_0(\theta, \phi, \varphi)\rangle = \cos \theta e^{i\phi} |0L\rangle + \sin \theta e^{i\varphi} |0R\rangle$ with a $U(2)$ coin $U(\alpha, \beta, \gamma)$. We discover that the average position $\bar{x} = \max(\bar{x}) \cos(\alpha + \gamma + \phi - \varphi)$, with coin operator $U(\alpha, \pi/4, \gamma)$ and initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = (e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)\sqrt{2}/2$. If we set initial state and coin operator to $|\Phi_0(\theta, \pi/2, 0)\rangle = i \cos \theta |0L\rangle + \sin \theta |0R\rangle$ and coin operator $U(0, \pi/4, 0)$, for $\alpha + \gamma + \phi - \varphi = \pi/2$, we discover that $\bar{x} = -\max(\bar{x}) \cos(2\theta)$. Last we verify the result above, and obtain the summarize properties of quantum walks with an arbitrary state. We get that $\bar{x}(\theta, \phi, \varphi, \alpha, \beta, \gamma, t) = \cos 2\theta * \bar{x}_{|0L\rangle}(\beta, t) + \sin 2\theta * \cos(\alpha + \gamma + \phi - \varphi) * \bar{x}_{(|0L\rangle+|0R\rangle)\sqrt{2}/2}(\alpha = \gamma = 0, \beta, t)$. If the average positions \bar{x} with initial state $|0L\rangle$ and $|\Psi_0\rangle = (|0L\rangle + |0R\rangle)\sqrt{2}/2$ and coin operator $U(0, \beta, 0)$ are known, we can get the average position result of quantum walks with an arbitrary initial state and a $U(2)$ coin operator.

Keywords: quantum walk, average position, initial state

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1 Introduction

Quantum walks (QWs) were first introduced in 1993 [1] as the quantum version of classical random walks. QWs can be divided into discrete time and continuous time [2] QWs. Both continuous time [3] and discrete time [4, 5] QWs can attain universal quantum computation. QWs have found widespread applications in quantum algorithms[6, 7, 8, 9, 10, 11]. In addition, QWs in graph [12], on a line with a moving boundary [13], with multiple coins [14], decoherent coins [15] or a $SU(2)$ coin [16, 18] have been discussed.

Here we discuss the average position properties for QWs with an arbitrary state and a $U(2)$ coin.

2 Discrete time quantum walks with an arbitrary initial state and a $U(2)$ coin operator

The total Hilbert space for discrete time QWs is given by $\mathcal{H} \equiv \mathcal{H}_P \otimes \mathcal{H}_C$, where \mathcal{H}_P is spanned by the orthonormal states $\{|x\rangle, x \in Z\}$ and \mathcal{H}_C is the two-dimensional coin space spanned by two orthonormal states $|L\rangle$ and $|R\rangle$.

In this paper, we discuss the QWs with an arbitrary initial state and a $U(2)$ coin operator. The probability distributions are the same with a $U(2)$ and a $SU(2)$ coin operator[18]. Here we use a $SU(2)$ matrix:

$$U(\alpha, \beta, \gamma) = \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix} \quad (1)$$

as the coin operator. The particle movement operator is given by

$$S = \sum_x (|x-1\rangle\langle x| \otimes |L\rangle\langle L| + |x+1\rangle\langle x| \otimes |R\rangle\langle R|) \quad (2)$$

After t steps QWs, the final state can be written as:

$$|\Psi_t\rangle = [SU(\alpha, \beta, \gamma)]^t |\Psi_0\rangle \quad (3)$$

where $|\Psi_0\rangle$ is an arbitrary initial state, is given by:

$$|\Psi_0(\theta, \phi, \varphi)\rangle = \cos \theta e^{i\phi} |0L\rangle + \sin \theta e^{i\varphi} |0R\rangle. \quad (4)$$

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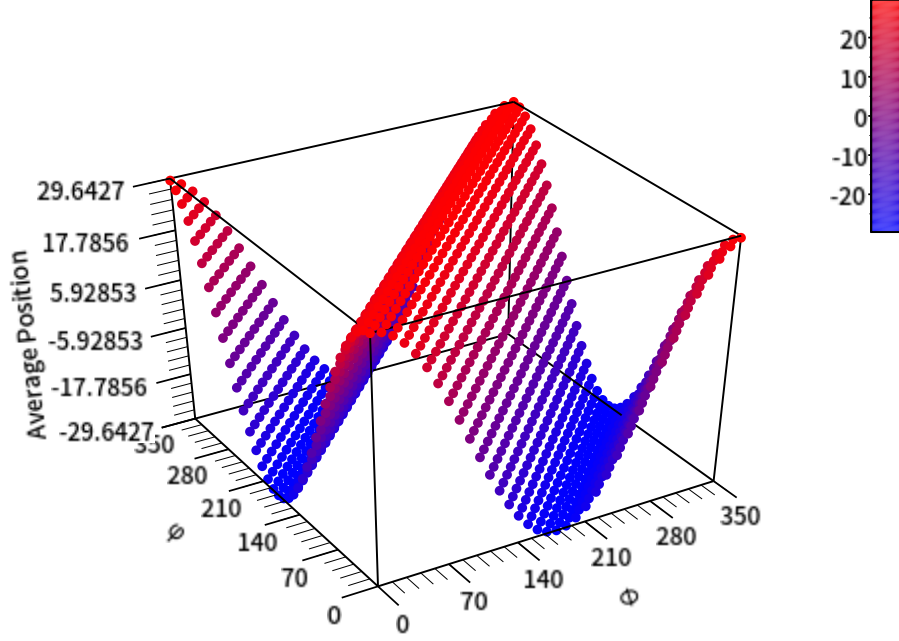


Figure 1: (Color online) The average position \bar{x} of quantum walks after $t = 100$ steps with SU(2) coin operator $U(0, \pi/4, 0)$ and initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$.

3 Average position of quantum walks with an arbitrary initial state

In this paper, we set the coin operator to $U(\alpha, \beta, \gamma)$, and set the initial state to $|\Phi_0(\theta, \phi, \varphi)\rangle$, then observe the change rule of average position with the changing of parameters $\alpha, \gamma, \theta, \phi$ and φ .

Figures 1 and 2 show the average position \bar{x} for QWs after 100 steps with a SU(2) coin $U(0, \pi/4, 0)$, while the initial state is $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$. From Fig. 1, we can know that \bar{x} only depends on $\phi - \varphi$. Fig. 2 shows that the actual \bar{x} exactly matches the function

$$f(\phi - \varphi) = \max(\bar{x}) \cos(\phi - \varphi). \quad (5)$$

Fig. 3 shows that the average position of QWs after 100 steps with different coin operators, but the same initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$. The two coin operators are $U(0, 45^\circ, 0)$ (black dot) and $U(52^\circ, 45^\circ, 77^\circ)$ (red line) respectively. From Fig. 3, we can know that the angle shift between the two peak values is $129^\circ = 52^\circ + 77^\circ$, it means that the curve of red line is left shift 129° to the curve of black dot in Fig. 3. Then we can conjecture that the average position of QWs with coin operator $U(\alpha, \pi/4, \gamma)$ and initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$ can be given by:

$$\bar{x} = \max(\bar{x}) \cos(\alpha + \gamma + \phi - \varphi). \quad (6)$$

Fig. 4 shows the average position of QWs after 100 steps with the changing of initial state $|\Phi_0\rangle(\theta, \pi/2, 0) = i \cos \theta |0L\rangle + \sin \theta |0R\rangle$ and coin operator $U(0, \pi/4, 0)$, Fig. 4 shows that the actual \bar{x} exactly matches the function

$$f(\theta) = -\max(\bar{x}) \cos(2\theta). \quad (7)$$

4 Proof in Mathematics

From the Ref. [18], we know that if the initial state $|\Psi_0(\theta, \phi, \varphi)\rangle = \cos \theta e^{i\phi} |0L\rangle + \sin \theta e^{i\varphi} |0R\rangle$, the coin operator

$$U(\alpha, \beta, \gamma) = \begin{pmatrix} e^{i\alpha} \cos \beta & -e^{-i\gamma} \sin \beta \\ e^{i\gamma} \sin \beta & e^{-i\alpha} \cos \beta \end{pmatrix}$$

the probability $P^L(x)$ at state $|xL\rangle$ and $P^R(x)$ at state $|xR\rangle$ after t steps of quantum walks can be given by

$$\begin{cases} P^L(x) = \cos^2 \theta P_{|0L\rangle}^L(x) + \sin^2 \theta P_{|0R\rangle}^L(x) - (e^{-i(\alpha+\gamma)} \cos \theta e^{-i\phi} \sin \theta e^{i\varphi} + e^{i(\alpha+\gamma)} \cos \theta e^{i\phi} \sin \theta e^{-i\varphi}) G^L(\beta, x, t) \\ P^R(x) = \cos^2 \theta P_{|0L\rangle}^R(x) + \sin^2 \theta P_{|0R\rangle}^R(x) - (e^{-i(\alpha+\gamma)} \cos \theta e^{-i\phi} \sin \theta e^{i\varphi} + e^{i(\alpha+\gamma)} \cos \theta e^{i\phi} \sin \theta e^{-i\varphi}) G^R(\beta, x, t) \end{cases}, \quad (8)$$

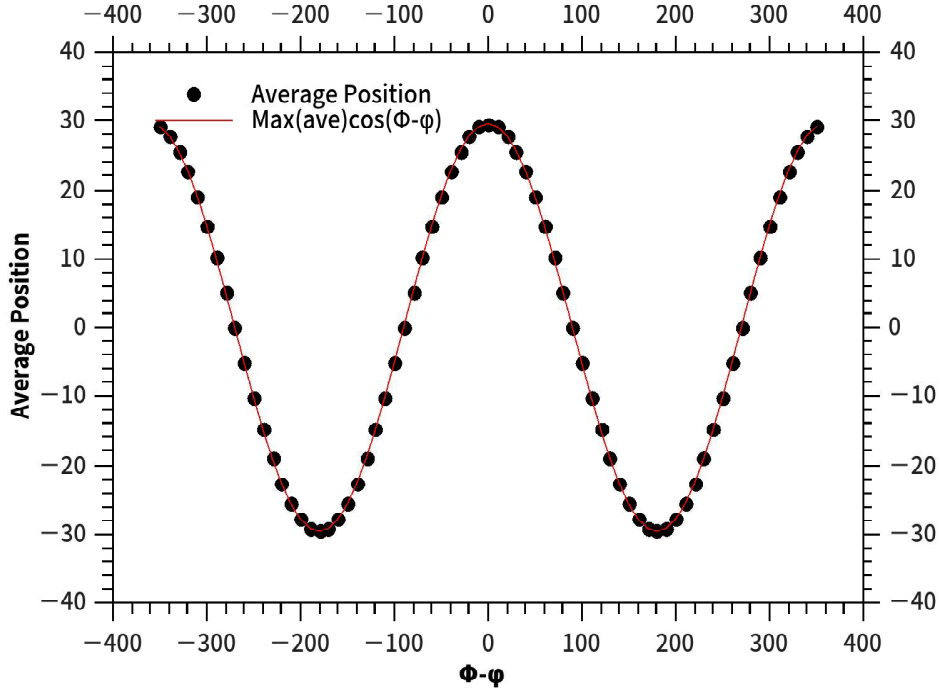


Figure 2: (Color online) (black dot) The average position \bar{x} with $\phi - \varphi$ after 100 steps quantum walk with SU(2) coin operator $U(0, \pi/4, 0)$ and initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$, (red line) function $f(\phi - \varphi) = \max(\bar{x}) \cos(\phi - \varphi)$.

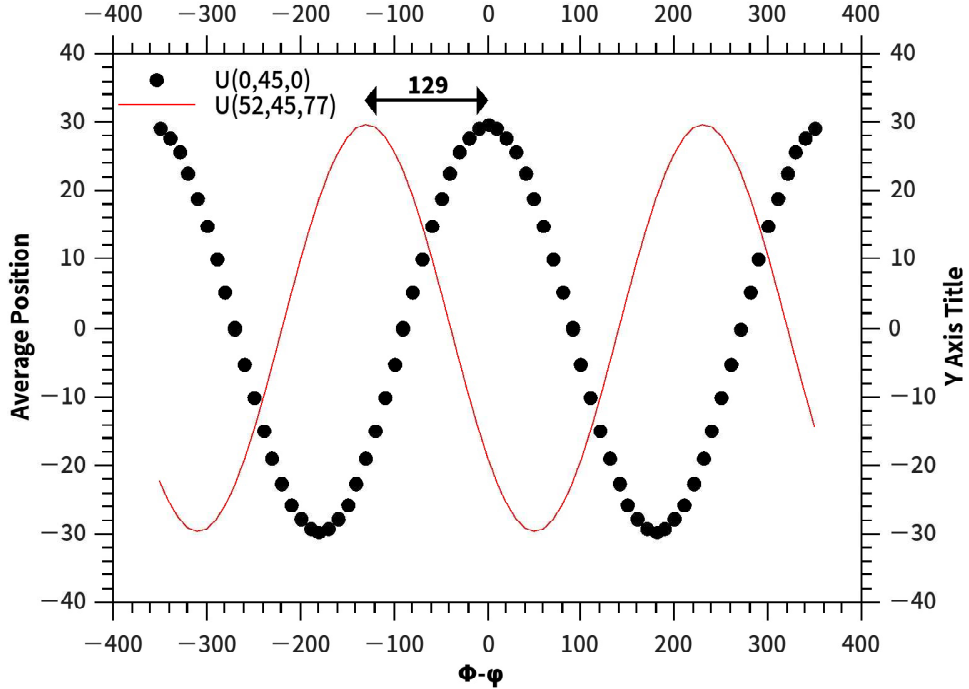


Figure 3: (Color online) The average position of quantum walks with initial state $|\Phi_0(\pi/4, \phi, \varphi)\rangle = \frac{\sqrt{2}}{2}(e^{i\phi} |0L\rangle + e^{i\varphi} |0R\rangle)$ after 100 steps while the coin operators are $U(0, 45^\circ, 0)$ (black dot) and $U(52^\circ, 45^\circ, 77^\circ)$ (red line). The angle shift between the two peak values is $129^\circ = 52^\circ + 77^\circ$.

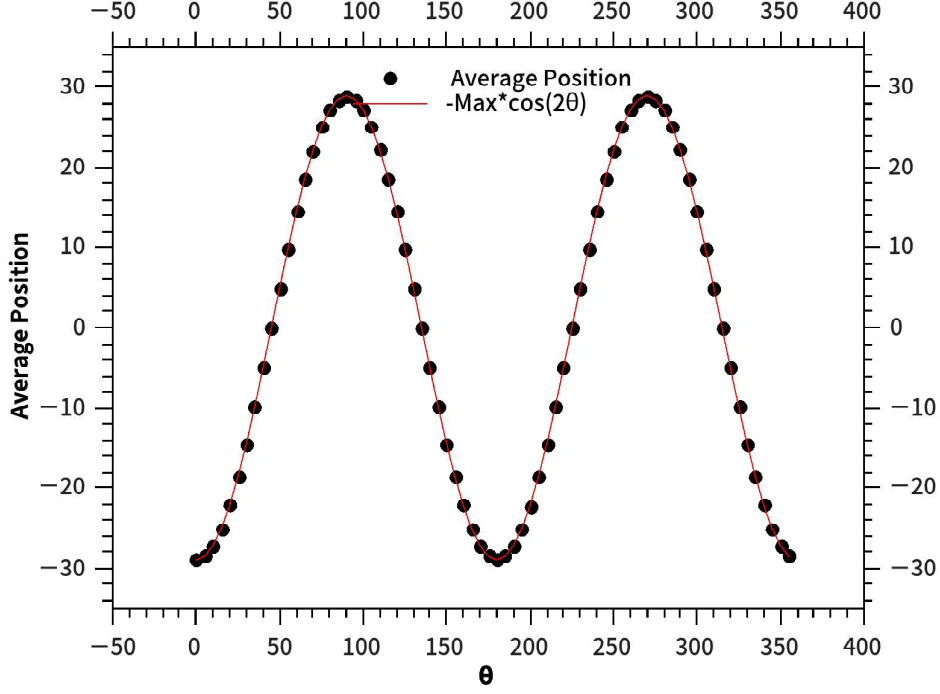


Figure 4: (Color online) (black dot) The average position of quantum walks after 100 steps with the changing of initial state $|\Phi_0\rangle(\theta, \pi/2, 0)$ and coin operator $U_C(0, \pi/4, 0)$, (red line) $f(\theta) = -\max(\bar{x})\cos(2\theta)$.

where $P_{|0L\rangle}^L(x)$ and $P_{|0L\rangle}^R(x)$ are the probability at state $|xL\rangle$ and $|xR\rangle$ with initial state $|0L\rangle$, $P_{|0R\rangle}^L(x)$ and $P_{|0R\rangle}^R(x)$ are the probability at state $|xL\rangle$ and $|xR\rangle$ with initial state $|0R\rangle$, $G^L(\beta, x, t)$ and $G^R(\beta, x, t)$ are functions only depend on angle β , position x and steps t .

After t steps, the average position of QWs $\bar{x} = \sum_{x=-t}^t x[P^L(\theta, \phi, \varphi, \alpha, \beta, \gamma, x, t) + P^R(\theta, \phi, \varphi, \alpha, \beta, \gamma, x, t)]$.

From Ref. [18], we know that $P_{|0j\rangle}^i(\alpha, \beta, \gamma, x, t) = P_{|0j\rangle}^i(\beta, x, t)$, $i, j \in \{L, R\}$ are irrelevant to the parameters α and γ , and $P_{|0L\rangle}^R(\beta, x, t) = P_{|0R\rangle}^L(\beta, -x, t)$, $P_{|0L\rangle}^L(\beta, x, t) = P_{|0R\rangle}^R(\beta, -x, t)$.

Then we can get that

$$\begin{cases} \sum_{x=-t}^t x[\cos^2 \theta P_{|0L\rangle}^L(\beta, x, t) + \sin^2 \theta P_{|0R\rangle}^R(\beta, x, t)] = \cos 2\theta \sum_{x=-t}^t x[P_{|0L\rangle}^L(\beta, x, t)] \\ \sum_{x=-t}^t x[\cos^2 \theta P_{|0L\rangle}^R(\beta, x, t) + \sin^2 \theta P_{|0L\rangle}^L(\beta, x, t)] = \cos 2\theta \sum_{x=-t}^t x[P_{|0L\rangle}^R(\beta, x, t)] \end{cases} \quad (9)$$

and

$$(e^{-i(\alpha+\gamma)} \cos \theta e^{-i\phi} \sin \theta e^{i\varphi} + e^{i(\alpha+\gamma)} \cos \theta e^{i\phi} \sin \theta e^{-i\varphi}) = \sin 2\theta * \cos(\alpha + \gamma + \phi - \varphi) \quad (10)$$

Using Eq. 9 and 10, we can get the average position

$$\bar{x} = \cos 2\theta * \bar{x}_{|0L\rangle}(\beta, t) + \sin 2\theta * \cos(\alpha + \gamma + \phi - \varphi) * G(\beta, t) \quad (11)$$

where $G(\beta, t) = -\sum_{x=-t}^t x[G^L(\beta, x, t) + G^R(\beta, x, t)]$.

If we set $\theta = \pi/4$, $\alpha + \gamma + \phi - \varphi = 0$, for easily, $\alpha = \gamma = \phi = \varphi = 0$, the coin operator is $U(0, \pi/4, 0) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$,

and the initial state is $|\Psi_0\rangle = \frac{\sqrt{2}}{2}(|0L\rangle + |0R\rangle)$, we can get $G(\beta, t) = \bar{x}(\theta = \pi/4, \alpha = \gamma = \phi = \varphi = 0, \beta, t)$.

Then Eq. 11 can be rewritten as

$$\bar{x}(\theta, \phi, \varphi, \alpha, \beta, \gamma, t) = \cos 2\theta * \bar{x}_{|0L\rangle}(\beta, t) + \sin 2\theta * \cos(\alpha + \gamma + \phi - \varphi) * \bar{x}_{\sqrt{2}(|0L\rangle + |0R\rangle)/2}(\alpha = \gamma = 0, \beta, t) \quad (12)$$

where $|0L\rangle$ and $\sqrt{2}(|0L\rangle + |0R\rangle)/2$ indicate the different initial states of QWs.

Using Eq. 12, we can get that $\bar{x}(\pi/4, \phi, \varphi, \alpha, \pi/4, \gamma, t) = \cos(\alpha + \gamma + \phi - \varphi) * \bar{x}_{\sqrt{2}(|0L\rangle + |0R\rangle)/2}(\alpha = \gamma = 0, \beta, t)$, this equation proves the results in Fig. 1, 2 and 3.

The same as above, we can get that $\bar{x}(\theta, \pi/2, 0, 0, \beta, 0, t) = \cos 2\theta * \bar{x}_{|0L\rangle}(\beta, t)$, this equation proves the result in Fig. 4. If we set $\bar{x}_{|0L\rangle}(\beta, t) = A$, $\cos(\alpha + \gamma + \phi - \varphi) * \bar{x}_{\sqrt{2}(|0L\rangle + |0R\rangle)/2}(\alpha = \gamma = 0, \beta, t) = B$, Eq. 12 can be rewritten as:

$$\bar{x}(\theta, \phi, \varphi, \alpha, \beta, \gamma, t) = \sqrt{A^2 + B^2} \cos(2\theta - \omega) \quad (13)$$

where $\cos \omega = \frac{A}{\sqrt{A^2+B^2}}$ is irrelevant to the parameter θ . The average position curve with θ still match the function of $\max(\bar{x}) \cos 2\theta$, but right shift $\omega/2$.

If we set the initial state to $|\Phi_0(\pi/4, 0, \pi/2)\rangle = 1/\sqrt{2}(|0L\rangle + i|0R\rangle)$, using Eq. 12 we can get that $\bar{x}(\pi/4, 0, \pi/2, \alpha, \beta, \gamma, t) = \sin(\alpha + \gamma) * \bar{x}_{\sqrt{2}(|0L\rangle + |0R\rangle)/2}(\alpha = \gamma = 0, \beta, t)$, this equation also proves the result in Ref. [18].

For any $\alpha + \gamma + \phi - \varphi = 0$, $\bar{x}(\theta = \pi/4, \alpha + \gamma + \phi - \varphi = 0, \beta, t) = \bar{x}(\pi/4, 0, 0, 0, \beta, 0, t)$.

Eq. 12 indicates the change rule of the average position in QWs with the parameters $\theta, \phi, \varphi, \alpha$ and γ . If we know average positions \bar{x} with coin operator $U(0, \beta, 0)$, while the initial states are $|0L\rangle$ and $\frac{\sqrt{2}}{2}(|0L\rangle + |0R\rangle)$, we can get the average position result for an arbitrary state using Eq. 12.

5 Conclusions

In this paper, we discussed the average position property in QWs with an arbitrary initial state $|\Psi_0\rangle(\theta, \phi, \varphi) = \cos \theta e^{i\phi} |0L\rangle + \sin \theta e^{i\varphi} |0R\rangle$ and a $U(2) = U(\alpha, \beta, \gamma)$ coin operator. Firstly, we set the coin operator and initial state to $U(0, \pi/4, 0)$ and $|\Phi_0(\pi/4, \phi, \varphi)\rangle$ respectively, we get that $\bar{x} = \max(\bar{x}) \cos(\phi - \varphi)$. Secondly, if we left the initial state unchanged, and set the coin operator to $U(\alpha, \pi/4, \gamma)$, we get that $\bar{x} = \max(\bar{x}) \cos(\alpha + \gamma + \phi - \varphi)$. Thirdly, we set the coin operator and initial state to $U(0, \pi/4, 0)$ and $|\Phi_0(\theta, \pi/2, 0)\rangle$ respectively, we get that $\bar{x} = -\max(\bar{x}) \cos(2\theta)$. Lastly, we poof the above result in mathematics, we get that the average position $\bar{x}(\theta, \phi, \varphi, \alpha, \beta, \gamma, t) = \cos 2\theta \bar{x}_{|0L\rangle}(\beta, t) + \sin 2\theta * \cos(\alpha + \gamma + \phi - \varphi) \bar{x}_{\sqrt{2}(|0L\rangle + |0R\rangle)/2}(\alpha = \gamma = 0, \beta, t)$. If we know average positions \bar{x} with coin operator $U(0, \beta, 0)$, while the initial states are $|0L\rangle$ and $|\Psi_0\rangle = \frac{\sqrt{2}}{2}(|0L\rangle + |0R\rangle)$, we can get the average position result of quantum walks for an arbitrary state and a $U(2)$ coin operator.

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